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ON THE STABILITY AND BIFURCATION OF THE STEADY STATE MOTIONS OF A HEAVY GYROSTAT*

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The Routh—Liapunov theorem /1/ and its inverse /2/ are used to investigate the stability and bifurcation of the steady state motions of a heavy gyrostat with a freely rotating rotor.

The equations of motion of a heavy gyrostat with a freely rotating rotor with one fixed point, admit the following first integrals:

$$U=\frac{1}{2}\Big[\sum_{(1\,2\,3)}(J_1\omega_1^2+2I\varphi^{\cdot}\omega_1\alpha_1+2\gamma_1e_1)+I\varphi^{\cdot 2}\Big]=\text{const} \tag{1}$$

$$U_1=\sum_{(1\,2\,3)}(J_1\omega_1+I\varphi^{\cdot}\alpha_1)\gamma_1=k=\text{const}$$

$$U_2=\varphi^{\cdot}+\sum_{(1\,2\,3)}\omega_1\alpha_1=\Omega=\text{const}$$

$$U_3=\sum_{(1\,2\,3)}\gamma_1^2=1$$

Here ω_i are the projections of the absolute angular velocity of the gyrostat on its principal axes of inertia x_i, J_i are the principal moments of inertia of the gyrostat, α_i are the cosines of the angles formed by the rotor axis with the x_i axes, I is the moment of inertia of the rotor relative to its axis of rotation, φ is the angle of rotation of the rotor, γ_i are cosines of the angles formed by the ascending vertical with the x_i axes, and e_i are constants proportional to the projections on the x_i axes of the vector drawn from the fixed point O to the center of mass C of the body. The summation sign with the subscript (123) means that the terms are obtained by the cyclic interchange of the indices.

We shall consider the steady state motions of the mechanical system in question, their stability and bifurcation. Let us introduce the function

$$W=U-\omega(U_1-k)+1/2\lambda\omega^2(U_3-1)-I_\mu(U_2-\Omega)$$

where ω,λ,μ are the undertermined Lagrange multipliers.

According to the Routh—Liapunov theorem, we have the following equations for determining the steady state motions of a system:

$$\partial W/\partial\omega_1=J_1\omega_1+g_1-\omega J_1\gamma_1-I\mu\alpha_1=0 \quad (1\ 2\ 3) \tag{2}$$

$$\frac{\partial W}{\partial\varphi^{\cdot}}=\sum_{(1\,2\,3)}I\omega_1\alpha_1+I\varphi^{\cdot}-\omega\sum_{(1\,2\,3)}I\alpha_1\gamma_1-I\mu=0$$

$$\frac{\partial W}{\partial\gamma_1}=e_1-\omega(J_1\omega_1+I\varphi^{\cdot}\alpha_1)+\lambda\omega^2\gamma_1=0 \quad (1\,2\,3)$$

where $g_i=I\varphi^{\cdot}\alpha_i$ are the projections of the gyrostatic moment vector on the x_i axes. From (2) we obtain

$$\omega_1=\omega\gamma_1,\quad \gamma_1=\frac{e_1-\omega g_1}{\omega^2(J_1-\lambda)}\quad (1\ 2\ 3),\quad \varphi^{\cdot}=\mu \tag{3}$$

Substituting these values into the integrals $U_1=k$, $U_2=\Omega$ and $U_3=1$, we obtain the following relations:

$$F(\omega,\lambda,\mu)=\omega^4-\sum_{(1\,2\,3)}\frac{(e_1-\omega g_1)^2}{(J_1-\lambda)^2}=0 \tag{4}$$

$$\Omega=\mu\Big[1-I\sum_{(1\,2\,3)}\frac{\alpha_1^2}{J_1-\lambda}\Big]+\sum_{(1\,2\,3)}\frac{e_1\alpha_1}{J_1-\lambda}$$

$$k=\frac{1}{\omega^3}\Big[\sum_{(1\,2\,3)}\frac{J_1e_1^2}{(J_1-\lambda)^2}-I\mu\omega\sum_{(1\,2\,3)}\frac{(J_1+\lambda)e_1\alpha_1}{(J_1-\lambda)^2}+\lambda I^2\mu^2\omega^2\sum_{(1\,2\,3)}\frac{\alpha_1^2}{(J_1-\lambda)^2}$$

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*Prikl.Matem.Mekhan.,45,No.3,572-575,1981 and (4) yield k, ω and μ as functions of λ and Ω. From (3) and (4) we conclude that for the given values of the parameters I, J_i, α_i, e_i $(i = 1, 2, 3)$, i.e. for a given mechanical system, the values of the quantities $\omega_1, \omega_2, \omega_3, \varphi^{\cdot}, \gamma_1, \gamma_2, \gamma_3$, corresponding to the steady state motions can be regarded as functions of two independent parameters λ and Ω. If the magnitude of the integral of the areas k is given, then the parameters λ and Ω will not be independent, but connected by the relation $k = k(\lambda, \Omega)$.

We shall study the stability of motion (3) with respect to $\varphi^{\cdot}, \omega_i, \gamma_i$ $(i = 1, 2, 3)$. We assume that in the perturbed motion

$$\omega_1 = \omega_1^{\circ} + \xi_1, \gamma_1 = \gamma_1^{\circ} + \eta_1 \ (1\ 2\ 3), \ \varphi^{\cdot} = \varphi^{\cdot\circ} + \zeta$$

The conditions of stability /3/ are obtained from the Rought—Liapunov theorem in the form of sufficient conditions of the sign determinacy of the second variation

$$\delta^2 W = IP\zeta^2 + \sum_{(1\,2\,3)} [J_1 z_1^2 - (J_1 - \lambda)\,\omega^2\eta_1] \tag{5}$$

$$z_1 = \xi_1 + \frac{I\alpha_1\zeta}{J_1} - \omega\eta_1 \ (1\ 2\ 3); \quad P = 1 - I\sum_{(1\,2\,3)} \frac{\alpha_1^2}{J_1}$$

on the linear manifold defined by the relations

$$\delta U_1 = \sum_{(1\,2\,3)} [J_1\gamma_1^{\circ} z_1 + (2J_1\gamma_1^{\circ}\omega + I\mu\alpha_1)\,\eta_1] = 0 \tag{6}$$

$$\delta U_2 = P\zeta + \sum_{(1\,2\,3)} (\alpha_1 z_1 + \omega\alpha_1\eta_1) = 0, \qquad \delta U_3 = \sum_{(1\,2\,3)} \gamma_1^{\circ}\eta_1 = 0$$

The necessary and sufficient conditions for the positive definiteness of the quadratic form (5) with conditions (6), are

$$\Delta_9 > 0, \ D = -\Delta_{10} > 0 \tag{7}$$

$$\Delta_9 = \frac{J_1J_2J_3I^2P}{\omega^2(J_1-\lambda)(J_2-\lambda)^2}\,[(J_2-\lambda)^3\alpha_1^2 + (J_1-\lambda)^3\alpha_2^2 - I(J_1-J_2)^2\alpha_1^2\alpha_2^2]\,[\lambda\omega^2 + (J_3-\lambda)\,\omega^2\gamma_3^2 + I\Omega(I-\Omega) - I^2\Omega^2(J_1-J_2)^2\alpha_1^2\alpha_2^2]$$

$$\frac{D}{J_1J_2J_3P\omega^2} = 4I\omega^2\sum_{(1\,2\,3)} J_1\gamma_1^2 - \sum_{(1\,2\,3)} (J_1-\lambda)\,\{A(\gamma_3\alpha_2 - \gamma_2\alpha_3)^2 + B(\gamma_3\alpha_2 - \gamma_2\alpha_3)(J_3-J_2)\,\gamma_2\gamma_3 + 4\omega^2\gamma_2^2\gamma_3^2(J_3-J_2)^2\} +$$

$$\omega^2\Big[\sum_{(1\,2\,3)} \gamma_1^2(J_2-\lambda)(J_3-\lambda)\Big]\Big[\sum_{(1\,2\,3)} J_1\gamma_1^2 - I\Big(\sum_{(1\,2\,3)} \gamma_1\alpha_1\Big)^2\Big]$$

$$A = I\Big(I\mu^2 - 2I\mu\omega\sum_{(1\,2\,3)} \gamma_1\alpha_1 + \omega^2\sum_{(1\,2\,3)} J_1\gamma_1^2\Big)$$

$$B = 4I\omega\Big(\omega\sum_{(1\,2\,3)} \gamma_1\alpha_1 - \mu\Big)$$

According to the sufficient conditions /2/ of invertibility of the Routh—Liapunov theorem, the second condition of (7) is also the necessary condition of stability of the motions (3). The latter assertion follows also from considering the characteristic polynomial for the equations in variations.

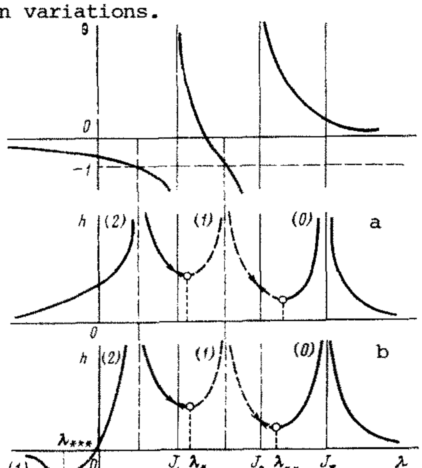

Fig.1

Substituting into the last equation of (2) the expressions for γ_i from (3), we obtain

$$F(\omega, \lambda, \mu) = \omega^4 - \sum_{(1\,2\,3)} \frac{(e_1 - \omega g_1)^2}{(J_1-\lambda)^2} = 0 \tag{8}$$

Let us consider a particular case when (8) is reduced to an equation biquadratic in ω. This requires that the following relations hold:

$$e_i g_i = 0 \ (i = 1, 2, 3) \tag{9}$$

When conditions (9) hold, the gyrostatic moment vector $g = (g_1, g_2, g_3)$ and the vector $e = (e_1, e_2, e_3)$ drawn from the fixed point O of the gyrostat to its center of mass are orthogonal, and lie in the principal planes of the inertia ellipsoid of the gyrostat relative to the point O. We shall assume for the definiteness that

$$e_1 = e_2 = 0, \ e_3 \neq 0$$

Then from the equation (8) we obtain

$$\omega^2 = \mu^2/2[R(\lambda) + Q(\lambda)]$$
$$\mu = \frac{\Omega}{1 - \theta(\lambda)}, \quad \theta(\lambda) = I \sum_{(1\,2)} \frac{\alpha_1^2}{\lambda - J_1}$$
$$R(\lambda) = I^2 \sum_{(1\,2)} \frac{\alpha_1^2}{(J_1 - \lambda)^2}, \quad Q(\lambda) = \left[R^2(\lambda) + \frac{4e_3^2}{\mu^4 (J_3 - \lambda)^2}\right]^{1/2}$$

In this case we obtain the following expression for the parameter k from (5):

$$k = \frac{1}{\omega^3}\left[\frac{J_3 e_3^3}{(J_3 - \lambda)^2} + I^2\mu^2\omega^2\lambda \sum_{(1\,2)} \frac{\alpha_1^2}{(J_1 - \lambda)^2}\right]$$

The expression for D is reduced to the form

$$D = -\Delta = \omega J_1 J_2 J_3 P \sigma(\lambda)[1 + \theta(\lambda)]\, Q(\lambda) \frac{dk}{d\lambda} \tag{10}$$
$$(\sigma(\lambda) = (J_1 - \lambda)(J_2 - \lambda)(J_3 - \lambda))$$

and the formula (10) makes it possible to relate the investigation of the condition of stability $D > 0$ to the analysis of the function $k = k(\lambda, \Omega)$.

Let us now consider the case

$$(J_2 - J_3)(J_3 - J_1)(J_1 - J_2)\,\alpha_1\alpha_2 \neq 0 \quad (J_1 < J_2 < J_3) \tag{11}$$

and study the distribution of the stable and unstable motions (3) on the surface $k = k(\lambda, \Omega)$. To do this, we shall consider the function $\theta(\lambda)$ and the some intersection of the surface $k = k(\lambda, \Omega)$ by the plane $\Omega = \Omega_0 \neq 0$. The roots of the equation $\theta(\lambda) = -1$ are real and equal to $\lambda = \lambda_1^*, \lambda_2^*$, and we also have $\lambda_1^* < J_1 < \lambda_2^* < J_2 < J_3$. We note that the values $\lambda_i^* (i = 1, 2)$ are independent of the parameter Ω.

Fig.1 depicts the form of one branch of the line of intersection of the surface $k = k(\lambda, \Omega)$ with the plane $\Omega = \Omega_0 \neq 0$ (the second branch can be obtained from the first branch by symmetric reflection in the abscissa), for the cases when a) conditions (4) and $J_3|e_3| > I^2\Omega^2$ hold and the equation $dk/d\lambda = 0$ has two real roots $\lambda = \lambda_*, \lambda_{**}$ $(J_1 < \lambda_* < J_2 < \lambda_{**} < J_3)$ and b) conditions (4) and $J_3|e_3| \cdot I^2\Omega^2$ hold and the equation $dk/d\lambda = 0$ has three real roots $\lambda = \lambda_*, \lambda_{**}, \lambda_{***}$ $(\lambda_{***} < J_1 < \lambda_* < J_2 < \lambda_{**} < \tau_3)$.

We note that the intersection of the surface $\omega = \omega(\lambda, \Omega)$ by the plane $\Omega = \Omega_0 \neq 0$ is analogous to that shown in Fig.1a.

Let $\lambda > J_3$. Then $1 + \theta(\lambda) > 0$, $\sigma(\lambda) < 0$, $dk/d\lambda < 0$ (Fig.1) and by virtue of (10) we have $D > 0$. Therefore the motion (3) is stable for $\lambda > J_3$. Similarly, if $\lambda_{**} < \lambda < J_3$, $J_1 < \lambda < \lambda_*$, $\lambda_1^* < \lambda < J_1$ or $\lambda < \lambda_1^*$, then $D > 0$. If $J_2 < \lambda < \lambda_{**}$ or $\lambda_2^* < \lambda < J_2$, then $D < 0$.

Proceeding in analogous manner we can establish the sign of D in the case when conditions (11) and $J_3|e_3| < I^2\Omega^2$ hold. The numbers (0), (1) and (2) in the Fig.1 indicate the degree of instability of the motions (3), and the values $\lambda = \lambda_*, \lambda_{**}$ or $\lambda = \lambda_*, \lambda_{**}, \lambda_{***}$ have the corresponding points of bifurcation at which the stability of the motions (3) is changed.

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